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b-MATCHING DEGREE-SEQUENCE POLYHEDRA

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A capacitated b-matching in a graph is an assignment of non-negative integers to edges, each at most a given capacity and the sum at each vertex at most a given bound. Its degree sequence is the vector whose components are the sums at each vertex. We give a linear-inequality description of the convex hull of degree sequences of capacitated b-matchings of a graph. This result includes as special cases theorems of Balas-Pulleyblank on matchable sets and Koren on degree sequences of simple graphs. We also give a min-max separation theorem, and describe a connection with submodular functions.

1. Introduction

A classical theorem of Erdős and Gallai [11] gives necessary and sufficient conditions for numbers $x_1, x_2, \ldots, x_n$ to be the degrees of the vertices of some simple n-vertex graph. The conditions may be described as some linear inequalities on the x_j together with the conditions that they be integers and have even sum. Every convex combination of degree sequences is also a solution of the linear inequalities. Koren [14] proved the converse; hence the convex hull of degree sequences is completely described by the Erdős-Gallai inequalities.

In a more recent paper Balas and Pulleyblank [1] considered in a graph G those subsets of vertices inducing a subgraph having a perfect matching. They gave a characterization by linear inequalities of the convex hull of incidence vectors of such "matchable" subsets. We provide a common generalization of the results of Koren and Balas-Pulleyblank. For a graph $G = (V, E)$ and positive integral vectors $b = (b_v : v \in V)$, $u = (u_e : e \in E)$ (we also allow $u_e = \infty$), a *(u-capacitated) b-matching* is a non-negative integral vector $z = (z_e : e \in E)$ such that $z \leq u$ and, for each $v \in V$, $\Sigma(z_e : e \text{ incident with } v) \leq b_v$. The *degree sequence of* z is $(x_v : v \in V)$ where x_v is the left-hand side of the latter inequality. Where G is complete, $|V| = n$, each $u_e = 1$ and each $b_v = n - 1$, the degree sequences of b-matchings are precisely the degree sequences in the sense of Erdős-Gallai. Where G is arbitrary, each $b_v = 1$ and each $u_e = \infty$, the degree sequences of b-matchings are precisely the incidence vectors of the matchable subsets of Balas-Pulleyblank.

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Here we characterize by linear inequalities the convex hull of degree sequences of capacitated b-matchings of any graph G. We actually prove a slightly stronger theorem, giving a min-max formula for the maximum amount by which a given point can violate an inequality of the system. The proof is by elementary constructions from the corresponding result for matchable subsets of a graph. We also derive the Koren and Balas-Pulleyblank results from our result, and give a short proof of the bipartite case. Finally, we sketch a second proof of the main result, based on submodularity.

In this paper we use standard graph terminology. For $x \in \mathbf{R}^S$ and $Q \subseteq S$, $x(Q)$ denotes $\Sigma(x_j : j \in Q)$. Where $G = (V, E)$ is a graph and A, B are subsets of V, we use $[A, B]$ to denote $\{e : e$ joins an element of A to an element of $B\}$. In particular, $\delta(A)$ denotes $[A, V \backslash A]$ and $\gamma(A)$ denotes $[A, A]$. We often omit brackets; for example, for $v \in V$, we may write $\delta(v)$ instead of $\delta(\{v\})$ and for $A, B \subseteq V$, $x(A, B)$ instead of $x([A, B])$.

The *degree sequence* of $z \in \mathbf{R}^E$ is the vector $x = Mz$, where M is the incidence matrix of G. (Clearly this agrees with the earlier definition when z is a b-matching.) Hence for any polyhedron $P \subseteq \mathbf{R}^E$ the set of degree sequences of elements of P is another polyhedron $P' \subseteq \mathbf{R}^V$. Moreover, an efficient algorithm to optimize an arbitrary linear objective function over P yields such an algorithm for P'. Namely, $\max(cx : x \in P') = \max((cM)z : z \in P)$. (Another basic problem for P' is the *separation* problem: Given $x \in \mathbf{R}^V$, find, if there is one, a linear inequality satisfied by every point of P', but violated by x. In contrast to the situation for optimization, there seems to be no such simple way to deduce the existence of an efficient separation algorithm for P' from one for P. However, there is a not-so-simple way, via the ellipsoid method [12].) Due to this close connection between P and P', one might expect that the existence of a nice description of P via linear inequalities would imply the existence of such a description for P'. We know of no such general result, and indeed there are a number of concrete unsolved problems of this kind. However, the class of polyhedra P for which P' is characterized in this paper *is* well understood; in fact, it is the subject of a famous result of Edmonds and Johnson [10].

2. Results

We begin by stating the result of Erdős-Gallai referred to in the introduction. This statement is easily seen to be equivalent to more usual versions.

Theorem 2.1. *$(x_v : v \in V)$ is the degree sequence of a simple graph having vertex-set V if and only if*

(a) x_v is an integer, for $v \in V$;

(b) $\sum_{v \in V} x_v$ is even;

(c) $x(A) - x(B) \leq |A|(|A| - 1) + |A|(|V| - |A| - |B|)$ for disjoint sets $A, B \subseteq V$;

(d) $x_v \geq 0$ for $v \in V$.

Therefore, Koren's result may be stated as follows.

Theorem 2.2. *The convex hull of degree sequences of simple graphs on vertex-set V is the solution set of (2.1c), (2.1d).*

Koren actually characterized the extreme points of the polyhedron. Recently, Peled and Srinivasan [17] have characterized its facets and proved that the system (2.1a), (2.1b) is totally dual integral. Next we state the result of Balas and Pulleyblank on 1-matchings.

Theorem 2.3. *The convex hull of degree sequences of 1-matchings of $G = (V, E)$ is the set of solutions of*
(a) $x_v \le 1$, for $v \in V$*;*
(b) $x\left(\bigcup_1^k A_i\right) - x(B) \le \sum_1^k (|A_i| - 1)$, *for all* $B \subseteq V$ *and vertex-sets* $A_1, \dots, A_k$ *of components of* $G - B$ *such that each* $|A_i|$ *is odd;*
(c) $x_v \ge 0$, *for* $v \in V$.

We remark that in (2.3b) $A_1, \dots, A_k$ need not be *all* such vertex-sets. Balas and Pulleyblank also proved that the system (2.3a), (2.3b), (2.3c) is totally dual integral and characterized the facets of the polyhedron. Our common generalization of Theorems 2.2 and 2.3 is the following.

Theorem 2.4. *The convex hull of degree sequences of u-capacitated b-matchings of* $G = (V, E)$ *is the set of solutions of*
(a) $x_v \le b_v$, *for* $v \in V$*;*
(b) $x\left(\bigcup_0^k A_i\right) - x(B) \le \sum_1^k (b(A_i) - 1) + u(\gamma(A_0)) + u(A_0, V \setminus B)$ *for disjoint subsets* B, A_0 *of* V *and vertex-sets* $A_1, \dots, A_k$ *of components of* $G - (B \cup A_0)$ *such that* $b(A_i) + u(A_0, A_i)$ *is odd,* $1 \le i \le k$*;*
(c) $x_v \ge 0$, $v \in V$.

We remark that if the right-hand side of (2.4b) is ∞, then of course that inequality may be considered not to be in the system. Let us show that Theorem 2.2 and 2.3 are consequences of Theorem 2.4. We do this first for Theorem 2.3. First observe that in this case (2.4a) and (2.4c) are the same as (2.3a) and (2.3c). Every inequality of (2.3b) is of the type (2.4b) by choosing $A_0 = \emptyset$; we have to show only that any other inequality of (2.4b) is redundant. Since each $u_e = \infty$, A_0 must be a set of vertices joined only to elements of B, so the same inequality is already present by considering each element of A_0 as a singleton A_i, $i = k + 1, \dots, k + |A_0|$, and then taking $A_0 = \emptyset$.

Next we derive Theorem 2.2 from Theorem 2.4. First, we observe that (2.1d) is the same as (2.4c) and that in this case (2.4a) is implied by (2.4b) with $k = 0$, $A_0 = \{v\}$, and $B = \emptyset$. Next, choosing $k = 0$ and denoting A_0 by A, the inequalities (2.1c) are included among the inequalities (2.4b). So we need only show that the remaining inequalities from (2.4b) are redundant. Since G is complete, the only other possibility is $k = 1$, when the right-hand side of (2.4b) is $q(n-1)-1+p(p-1)/2+p(n-r)$, where $|A_0| = p$, $|A_1| = q$, and $|B| = r$. But (2.1c) with $A = A_0 \cup A_1$ gives the same left-hand side and right-hand side $(p+q)(n-r-1) = q(n-1)+p(n-r)-qr-p$, which is larger only if $p = qr = 0$. But then $b(A_1) + u(A_0, A_1) = q(n-1) = q(q+r-1) = q(q-1)$, which is even, so this is impossible.

Let us mention one more special case of Theorem 2.4, namely, that for bipartite graphs, treated for 1-matching in [2]. In its statement, $N(A)$ denotes the neighbour set of A, that is, $N(A) = \{v \in V : uv \in E \text{ for some } u \in A\}$.

Theorem 2.5. *Let $G = (V, E)$ be bipartite with bipartition $\{Q, R\}$. The convex hull of degree sequences of u-capacitated b-matchings of G is the set of solutions of*
(a) $x_v \le b_v$, for $v \in V$;
(b) $x(Q) - x(R) = 0$;
(c) $x(A) - x(B) \le u(A, N(A) \setminus B)$, *for* $A \subseteq Q$, $B \subseteq R$;
(d) $x_v \ge 0$, *for* $v \in V$.

We leave to the reader the exercise of deriving Theorem 2.5 from Theorem 2.4. However, Theorem 2.5 is also an easy consequence of well-known network flow results. Namely, x is in the degree-sequence polyhedron if and only if there exists z in the u-capacitated b-matching polyhedron such that $x = Mz$, that is, such that

$$0 \le z_e \le u_e, \text{ for } e \in E;$$
$$x_v = z(\delta(v)) \le b_v, \text{ for } v \in V.$$

Therefore x is in the degree-sequence polyhedron if and only if $0 \le x \le b$ and the following system has a solution:

$$0 \le z_e \le u_e, \text{ for } e \in E;$$
$$z(\delta(v)) = x_v, \text{ for } v \in V.$$

The well-known necessary and sufficient conditions for this are precisely (2.5bc). It also follows that membership of a vector in this polyhedron can be tested by solving a maximum flow problem. For the case of 1-matching this is already known; see [16].

Theorem 2.4, which characterizes a degree-sequence polyhedron in terms of linear inequalities, can be strengthened into a min-max separation theorem. Let us illustrate this idea first for Theorem 2.3. Let x be any vector in $\mathbf{R}^V$ satisfying $0 \le x \le 1$. Consider the problem $\max(y(V) : y \le x,\ y \in P(G))$, where $P(G)$ denotes the convex hull of degree sequences of 1-matchings of G. Obviously, the maximum is $x(V)$ if and only if $x \in P(G)$. Now let $B \subseteq V$ and $A_1, A_2, \ldots, A_k$ be vertex-sets of odd components of $G - B$. Then for any $y \in P(G)$ with $y \le x$, we have

$$\begin{aligned} y(V) &= y\Big(\bigcup_1^k A_i\Big) - y(B) + y(B) + y\Big(V \setminus \bigcup_1^k A_i\Big) \\ &\le \sum_1^k (|A_i| - 1) + x(B) + x\Big(V \setminus \bigcup_1^k A_i\Big). \end{aligned}$$

This verifies the easy part of the following theorem.

Theorem 2.6. *Let $P(G)$ be the 1-matching degree-sequence polyhedron of G, and let $x \in \mathbf{R}^V$ with $0 \le x \le 1$. Then*

$$\max(y(V) : y \le x,\ y \in P(G)) = \min\Big(\sum_1^k (|A_i| - 1) + x(B) + x\Big(V \setminus \bigcup_1^k A_i\Big)\Big),$$

where the minimum is over subsets B of V and vertex-sets $A_1, \ldots, A_k$ of odd components of $G - B$.

Most of the interest of Theorem 2.6 arises from its connection with the separation problem for the constraints (2.3b). Namely, the minimum is $x(V) + \min\left(\sum_1^k(|A_i| - 1) - x\left(\bigcup_1^k A_i\right) + x(B)\right)$, that is, except for the constant $x(V)$, it gives the maximum amount by which x violates a constraint (2.3b). (In addition, $x \in P(G)$ if and only if this amount is ≥ 0, that is, Theorem 2.3 is an immediate consequence of Theorem 2.6.) Theorem 2.6 was proved in [6] by a polynomial-time algorithm that terminates with a maximizing y and minimizing $B, A_1, \ldots, A_k$. We shall use Theorem 2.6 to prove its analogue for capacitated b-matching, Theorem 2.7 below, from which Theorem 2.4 follows.

Theorem 2.7. *Let $P = P(G,b,u)$ denote the degree-sequence polyhedron of u-capacitated b-matchings of $G = (V,E)$, and let $x \in \mathbf{R}^V$, $0 \leq x \leq b$. Then*

$$\max(y(V) : y \leq x,\ y \in P) =$$
$$\min\left(\sum_1^k(b(A_i) - 1) + u(\gamma(A_0)) + u(A_0, V \setminus B) + x(B) + x\left(V \setminus \bigcup_0^k A_i\right)\right),$$

where the minimum is over disjoint subsets B, A_0 of G and vertex-sets $A_1, \ldots, A_k$ of components of $G - (A_0 \cup B)$ such that $b(A_i) + u(A_i, A_0)$ is odd, $1 \leq i \leq k$.

3. Proofs

The proof of Theorem 2.7 from Theorem 2.6 proceeds through a single intermediate result, the special case of Theorem 2.7 where the u_e are all infinity.

Theorem 3.1. *Let $P = P(G,b)$ denote the degree-sequence polyhedron of (uncapacitated) b-matchings of $G = (V,E)$, and let $x \in \mathbf{R}^V$, $0 \leq x \leq b$. Then*

$$\max(y(V) : y \leq x,\ y \in P) = \min\left(\sum_1^k(b(A_i) - 1) + x(B) + x\left(V \setminus \bigcup_0^k A_i\right)\right),$$

where the minimum is over disjoint subsets $B, A_0, A_1, \ldots, A_k$ of V such that A_0 is a set of isolated vertices of $G - B$ and $A_1, \ldots, A_k$ are vertex-sets of components of $G - B$ such that $b(A_i)$ is odd, $1 \leq i \leq k$.

(We remark that a corollary is that the b-matching degree-sequence polyhedron is given by $0 \leq x \leq b$ and $x\left(\bigcup_0^k A_i\right) - x(B) \leq \sum_1^k(b(A_i) - 1)$.)

Proof. To see that max $\leq$ min, we observe that, for any b-matching z with degree-sequence y and $1 \leq i \leq k$, at least one of $y(A_i) \leq b(A_i) - 1$ or $z(A_i, B) \geq 1$ is true, which implies $y(A_i) \leq b(A_i) - 1 + z(A_i, B)$. Adding these inequalities for $1 \leq i \leq k$, and adding also the inequality $y(A_0) \leq z(A_0, B)$, we get $y\left(\bigcup_0^k A_i\right) \leq$

$\sum_1^k(b(A_i)-1)+z\left(\bigcup_0^k A_i, B\right)$. But clearly the last term is at most $y(B)$. Hence $y\left(\bigcup_0^k A_i\right)-y(B)\le\sum_1^k(b(A_i)-1)$ holds for every $y\in P$; it follows that, if $y\le x$, then max $\le$ min.

Now to complete the proof we need to exhibit $y, B, A_0, A_1, \dots, A_k$ giving equality. We construct from G a new graph $G'=(V',E')$ as follows. (This construction is standard.) Each $v\in V$ has a set $C(v)=\{v_1,\dots,v_{b_v}\}$ of copies in V'. We define $v_iw_j\in E'$ if and only if $vw\in E$, and define $x'_{v_j}=x_v/b_v$. Then $0\le x'\le 1$, and we apply Theorem 2.6 to obtain y', a convex combination of degree sequences of matchings z' of G' and $B', A'_1, A'_2, \dots A'_m\subseteq V'$. But each matching z' of G' defines a b-matching z of G in a natural way: where $e=vw\in E$, z_e is the sum of z'_j over edges j of G' joining a copy of v to a copy of w. Moreover, the same convex combination y of the degree sequences of these b-matchings z will satisfy $y_v\le b_vx'_v=x_v$, and similarly $y(V)=\sum_{v\in V}b_vy'_v=y'(V')$. Hence we need to find $B, A_0, A_1, \dots, A_k$ in G giving

$$\begin{aligned}&\sum_1^k(b(A_i)-1)+x(B)+x\left(V\setminus\bigcup_0^k A_i\right)\\ =\ &\sum_1^m(|A'_i|-1)+x'(B')+x'\left(V'\setminus\bigcup_1^m A'_i\right).\end{aligned}$$

Let f represent the last quantity. We shall show that $B', A'_1, \dots, A'_m$ can be required to take a special form. Namely, we claim that there exist minimizing $B', A'_1, \dots, A'_m$ such that, for each $v\in V$, $C(v)\subseteq B'$, or $C(v)\subseteq A'_\ell$ for some ℓ, or each element of $C(v)$ is a singleton A'_ℓ, or $C(v)\cap\left(\bigcup_0^m A'_i\cup B'\right)=\emptyset$. Suppose that the claim is true, and renumber the A'_i as $A'_1,\dots,A'_k,A'_{k+1},\dots,A'_m$ where $|A'_i|=1$ if and only if $k+1\le i\le m$. Now put $B=\{v\in V: C(v)\subseteq B'\}$; put $A_i=\{v\in V: C(v)\subseteq A'_i\}$, $1\le i\le k$; put $A_0=\{v\in V:\{v_j\}=A'_i \text{ for some } i>k\}$. Then $B, A_0, A_1, \dots, A_k$ do have the required properties, and $f=x(B)+x\left(V\setminus\bigcup_0^k A_i\right)+\sum_1^k(b(A_i)-1)$, as required.

It remains to prove the claim. Suppose, for some $v\in V$ and some ℓ, $1\le\ell\le m$, that $v_j\in A'_\ell$, $v_k\notin A'_\ell$. Then every neighbour of v_j in G' is either in A'_ℓ or in B'. If every neighbour is in B', then each copy of v is a singleton component of $G'-B'$, and making each copy a singleton A'_i, will not increase f. Hence we may assume that there is at least one neighbour of v_j in A'_ℓ. Then necessarily $v_k\in B'$. Now if we discard the set A'_ℓ and delete v_k from B', then f decreases by $|A'_\ell|-1-x'(A'_\ell)+x'_{v_k}=|A'_\ell\setminus\{v_j\}|-x'(A'_\ell\setminus\{v_j\})\ge 0$, since $x'\le 1$. ∎

Proof of Theorem 2.7. As before, it is easy to derive max $\le$ min from the fact that each inequality (2.4b) is valid for P. To show the latter fact, let z be a u-capacitated

b-matching of G having degree sequence y. Then for $1 \le i \le k$, at least one of the inequalities $y(A_i) \le b(A_i) - 1$, $z(A_i, B) \ge 1$, $z(A_0, A_i) < u(A_0, A_i)$ must be satisfied. This fact implies the inequality

$$b(A_i) - y(A_i) + z(A_i, B) + u(A_0, A_i) - z(A_0, A_i) \ge 1.$$

Adding these inequalities gives

$$y\left(\bigcup_1^k A_i\right) \le \sum_1^k (b(A_i) - 1) + z\left(\bigcup_1^k A_i, B\right) + u\left(\bigcup_1^k A_i, A_0\right) - z\left(\bigcup_1^k A_i, A_0\right).$$

If we add also the inequality

$$y(A_0) \le z(A_0, B) + z\left(\bigcup_1^k A_i, A_0\right) + u(\gamma(A_0)) + u\left(A_0, V \setminus B \setminus \bigcup_1^k A_i\right)$$

and use the fact that $z(A_0, B) + z\left(\bigcup_1^k A_i, B\right) \le y(B)$, we get the desired inequality.

Now we must show that there exist $y, B, A_0, A_1, \ldots, A_k$ giving equality in the formula. Again we use a standard construction, forming a graph $G' = (V', E')$ from G by replacing each edge $e = vw$ of G by two vertices $v(e), w(e)$ with v adjacent $v(e)$, $v(e)$ adjacent $w(e)$, and $w(e)$ adjacent w. We define $b' = (b'_t : t \in V')$ and $x' = (x'_t : t \in V')$ by $b'_v = b_v$ and $x'_v = x_v$ if $v \in V$, and $b'_{v(e)} = x'_{v(e)} = u_e$ for $v \in V$ incident in G to $e \in E$. Then $0 \le x' \le b'$ and we apply Theorem 3.1 to obtain y', a convex combination of degree sequences of b'-matchings z' of G', and $B', A'_0, A'_1, \ldots, A'_m \subseteq V'$. We can assume that for any such z' and any $e = vw \in E$, the edges $e_1 = vv(e)$, $e_2 = v(e)w(e)$, and $e_3 = w(e)w$ satisfy $z'_{e_1} = z'_{e_3} = u_e - z'_{e_2}$. The reason is that z'_{e_1}, z'_{e_3} may be replaced by their minimum and z'_{e_2} defined so that this is satisfied; the modification does not change the facts that z' is a b'-matching and that $y' \le x'$, and it does not decrease $y'(V')$. Hence we may assume that $y'_{e(v)} = x'_{e(v)} = b'_{e(v)}$ for all v, e. Now every such z' defines a u-capacitated b-matching z of G by $z_e = z'_{e_1}$. Moreover, the same convex combination y of these z satisfies $y \le x$ and $y'(V') = y(V) + 2u(E)$. Therefore to complete the proof, we need to define $B, A_0, A_1, \ldots, A_k \subseteq V$ as in the theorem with

$$\sum_1^k (b(A_i) - 1) + u(\gamma(A_0)) + u(A_0, V \setminus B) + x(B) + x\left(V \setminus \bigcup_0^k A_i\right) + 2u(E)$$
$$= \sum_1^m (b'(A'_i) - 1) + x'(B') + x'\left(V' \setminus \bigcup_0^m A'_i\right).$$

We denote the last expression by f.

Again we show that $B', A'_0, \ldots, A'_m$ can be assumed to take a special form. In the following we use repeatedly the observation that for $1 \le i \le m$, $x'(A'_i) \ge b'(A'_i) - 1$, since otherwise discarding A'_i would decrease f.

Claim. If $v(e) \in B'$, we may assume that $v \in A'_0$. Suppose the claim is false. We shall delete $v(e)$ from B'. Suppose first that $\left|\{v, w(e)\} \cap \bigcup_0^m A'_i\right| \leq 1$. If it is 0, obviously f decreases. If it is 1, we remove the corresponding A'_p or element of A'_0. In the former case, f goes down by at least $x'_{v(e)} - 1 = u_e - 1 \geq 0$. In the latter case, we have $w(e) \in A'_0$, and f goes down by exactly $x'_{v(e)} - x'_{w(e)} = 0$. Now suppose that $w(e) \in A'_0$ and $v \in A'_p$ for some $p > 0$. Adding both $v(e)$ and $w(e)$ to A'_p, f goes down by

$$2x'_{v(e)} - b'_{v(e)} - b'_{w(e)} = 2u_e - 2u_e = 0.$$

Finally, suppose that $v(e) \in A'_p$, $w(e) \in A'_q$ with $p, q > 0$ and possibly $p = q$. If we discard both A'_p, A'_q, then f goes down by at least $z'_{v(e)} - |\{p, q\}| = u_e - |\{p, q\}|$. So we need only deal with the case when $u_e = 1$ and $p \neq q$. But then we can replace A'_p, A'_q by $A'_p \cup A'_q \cup \{v(e)\}$, and f goes down by at least $2z'_{v(e)} - 1 = 2u_e - 1 > 0$. This completes the proof of the claim.

Now define $k = m$, $B = B' \cap V$, and $A_i = A'_i \cap V$ for $0 \leq i \leq k$. It follows from the claim that if $vw \in E$ and $v \in A_p$, $p \geq 1$, then $w \in A_p \cup A_0 \cup B$. Hence A_p is the vertex-set of a component of $G - (B \cup A_0)$, provided that $A_p \neq \emptyset$. But if A_p is empty, then $A'_p \subseteq \{v(e), w(e)\}$ for some $e = uv \in E$. Equality cannot hold, because $b'(A'_p)$ is odd. Moreover, if $A'_p = \{v(e)\}$, then discarding A'_p and adding $v(e)$ to A'_0 does not increase f. Now $v, v(e) \in A'_p$ implies $w(e) \in A'_p$ and $w \in B' \cup A'_p$, or $w(e) \in B'$ and $w \in A'_0$. Hence $b'(A'_p) = b(A_p) + 2u(A_p, B) + 2u(\gamma(A_p)) + u(A_p, A_0)$, so $b(A_p) + u(A_p, A_0)$ is an odd integer. Using the claim, we can easily classify all of the edges vw of G as follows. Here we denote $(V' \setminus B') \setminus \bigcup_0^k A'_i$ by C' and $C' \cap V$ by C. If

$v, w \in A_p,\ p \geq 1$	then	$v(e),\ w(e) \in A'_p$;
$v \in A_p,\ p \geq 1, w \in A_0,$	then	$v(e) \in A'_p,\ w(e) \in B'$;
$v \in A_p,\ p \geq 1, w \in B,$	then	$v(e), w(e) \in A'_p$;
$v, w \in A_0,$	then	$v(e), w(e) \in B'$;
$v \in A_0,\ w \in B,$	then	$v(e) \in B',\ w(e) \in A'_0$;
$v \in A_0,\ w \in C,$	then	$v(e) \in B',\ w(e) \in C'$;
$v, w \in B,$	then	$v(e), w(e) \in C'$;
$v \in B,\ w \in C,$	then	$v(e),\ w(e) \in C'$;
$v, w \in C,$	then	$v(e), w(e) \in C'$.

We can now compute $x'(B')$ to be $x(B) + u\left(A_0, \bigcup_1^k A_i\right) + 2u(\gamma(A_0)) + u(A_0, B) + u(A_0, C)$. Similarly, $x'\left(V' \setminus \bigcup_0^k A'_i\right) = x'(B') + x'(C')$, and $x'(C') = x(C) + u(A_0, C) +$

$2u(\gamma(B)) + 2u(B,C) + 2u(\gamma(C))$. Now a calculation shows that

$$\sum_{i=1}^{k}(b'(A'_i) - 1) + x'(B') + x'\left(V' \setminus \bigcup_{0}^{k} A'_i\right) - 2u(E)$$
$$= \sum_{i=1}^{k}(b(A_i)-1)+u(\gamma(A_0))+u(A_0, V \setminus B)+x(B)+x\left(V \setminus \bigcup_{0}^{k} A_i\right),$$

as required. ∎

4. Submodularity

Recent results on generalizations of matroids, polymatroids, and submodularity lead to a different approach to the results of this paper. In this section we outline this approach.

The matchable sets of a graph form the "feasible sets" of a structure variously called a "metroid" (Dress and Havel [7]), a "pseudomatroid" (Chandrasekaran and Kabadi [5]), and a "delta-matroid" (Bouchet [3]). (Actually, delta-matroids and pseudomatroids are identical, but metroids satisfy an additional "normalizing" condition, that $\emptyset$ is feasible.) Bouchet has given a single axiom to characterize the family $\mathcal{F}$ of feasible sets of a delta-matroid:

(4.1) If $F_1, F_2 \in \mathcal{F}$ and $a \in F_1 \triangle F_2$, then there exists $b \in F_1 \triangle F_2$ such that $F_1 \triangle \{a, b\} \in \mathcal{F}$.

(Here $\triangle$ denotes symmetric difference.) It is known [4], [5] that the convex hull of incidence vectors of feasible sets of a delta-matroid on a set V is described by

$$x(A) - x(B) \leq f(A, B), \qquad A, B \subseteq V,\ A \cap B = \emptyset. \tag{4.2}$$

Here $f(A, B)$ denotes $\max(|F \cap A| - |F \cap B| : F \in \mathcal{F})$. The proof follows a greedy algorithm [3], [5] for optimizing any linear function over the feasible sets. (This type of greedy algorithm was first used by Dunstan and Welsh [8].) From this one can already conclude that the matchable set polyhedron is given by inequalities having coefficients $0, 1, -1$. Using well-known results on matching, for example, Edmonds' algorithm, one can obtain a formula for $f(A, B)$, and deduce the Balas-Pulleyblank Theorem 2.3. This was pointed out independently by A. Bouchet (private communication).

The rank function f introduced above obviously is *normalized*, that is, $f(\emptyset, \emptyset) = 0$; less obviously, it is *bisubmodular*, that is, for all $A, B, A', B' \subseteq V$,

$$f(A, B) + f(A', B') \geq f(A \cap A', B \cap B') + f((A \cup A') \setminus (B \cup B'), (B \cup B') \setminus (A \cup A')). \tag{4.3}$$

Kabadi and Chandrasekaran [13], Nakamura [15], and Qi [18] have shown that for any normalized, bisubmodular f, the system (4.2) is totally dual integral. (Nakamura's result is stated in a slightly weaker form.) Hence the polyhedron $P(f)$ of solutions to (4.2) is the convex hull of the set of integral solutions, if f is integer-valued. We show that a min-max result like Theorem 2.6 also holds. For its proof we need to describe one more basic result on bisubmodularity.

Where $c \in \mathbf{R}^V$ and f is bisubmodular and normalized, consider the dual pair of linear programs

$$\text{(4.4)}\qquad \begin{aligned} &\text{maximize } \textstyle\sum(c_j y_j : \ j \in V) \\ &\quad \text{subject to} \\ &y(A) - y(B) \leq f(A,B), \qquad A, B \subseteq V, \quad A \cap B = \emptyset. \end{aligned}$$

$$\text{(4.5)}\qquad \begin{aligned} &\text{minimize } \textstyle\sum(f(A,B) s_{A,B} : \ A, B \subseteq V, \ A \cap B = \emptyset) \\ &\quad \text{subject to} \\ &\textstyle\sum(s_{A,B} : \ j \in A) - \sum(s_{A,B} : \ j \in B) = c_j, \qquad j \in V; \\ &s_{A,B} \geq 0, \ A, B \subseteq V, \ A \cap B = \emptyset. \end{aligned}$$

The (primal) *greedy algorithm* for solving (4.4) is: Order V as $\{1, 2, \ldots, n\}$ so that $|c_1| \geq |c_2| \geq \ldots \geq |c_n|$; for $0 \leq k \leq n$, let $A_k = \{i : \ i \leq k, \ c_i \geq 0\}$, and let $B_k = \{i : \ i \leq k, \ c_i < 0\}$; for $1 \leq k \leq n$, let $y_k = f(A_k, B_k) - f(A_{k-1}, B_{k-1})$ if $c_k \geq 0$, and let $y_k = -f(A_k, B_k) + f(A_{k-1}, B_{k-1})$ if $c_k < 0$. The *dual greedy algorithm* for solving (4.5) is: Put $s_{A_k, B_k} = |c_{k+1}| - |c_k|$ for $1 \leq k < n$ and put $s_{A_n, B_n} = |c_n|$. It is not too difficult to prove that these algorithms do deliver optimal solutions of (4.4) and (4.5); see [13] or [15] for details. Notice that the fact that $max(y(A) - y(B) : \ y \in P(f)) = f(A, B)$ is a consequence of the correctness of the two algorithms.

Theorem 4.6. *If f is bisubmodular and normalized, if $x \in \mathbf{R}^V$, and if there exists $y \in P(f)$ with $y \leq x$, then*

$$\max(y(V) : \ y \leq x, \ y \in P(f)) = \min(f(A,B) + x(B) + x(V \setminus A) : A, B \subseteq V, A \cap B = \emptyset).$$

Proof. We use a technique introduced by Edmonds [9]. Consider the dual pair of linear programs

$$\text{(4.7)}\qquad \begin{aligned} &\text{maximize } \textstyle\sum(y_j : \ j \in V) \\ &\quad \text{subject to} \\ &y(A) - y(B) \leq f(A,B), \qquad A, B \subseteq V, \quad A \cap B = \emptyset; \\ &y_j \leq x_j, \qquad j \in V. \end{aligned}$$

$$\text{(4.8)}\qquad \begin{aligned} &\text{minimize } \textstyle\sum(f(A,B) s_{A,B} : \ A, B \subseteq V, \ A \cap B = \emptyset) + \sum(x_j t_j : \ j \in V) \\ &\text{subject to} \\ &\textstyle\sum(s_{A,B} : \ j \in A) - \sum(s_{A,B} : \ j \in B) + t_j = 1, \qquad j \in V; \\ &s_{A,B} \geq 0, \qquad A, B \subseteq V, \ A \cap B = \emptyset; \\ &t_j \geq 0, \qquad j \in V. \end{aligned}$$

The optimal value of (4.7) is clearly the left-hand side of the max-min relation of the theorem. We shall show that the optimal value of (4.8) is the right-hand side, so that the theorem will follow from the duality theorem. Suppose that the values of t in (4.8) have been chosen. Then the problem of finding values for s is a problem of the form (4.5) with $c_j = 1-t_j$, $j \in V$, and so can be solved by the dual greedy algorithm. Therefore, there is an optimal solution to (4.8) whose positive dual variables induce a submatrix of the constraint matrix that, after some row and column scaling by -1, is the incidence matrix of a chain of sets, with an identity matrix appended. Such a matrix is easily seen (and well known) to be totally unimodular. Therefore, (4.8) has an optimal solution that is integer-valued. Moreover, by fixing the optimal integral values for t, we can apply the above argument to conclude that there is an optimal integer-valued solution (s, t), such that s is found by the dual greedy algorithm with weights $1 - t_j$.

Next we prove that t can be chosen to be $\{0, 1, 2\}$-valued. Suppose that some $t_j > 2$. After V has been ordered for the greedy algorithm, choose k maximal so that $|c_1| = |c_2| = \ldots = |c_k|$. Notice that $A_k = \emptyset$. If we lower s_{A_k,B_k} by 1 and also lower $t_1, t_2, \ldots, t_k$ by 1, then the new solution is feasible to (4.8), and the objective function decreases by $x(B_k) + f(\emptyset, B_k) = x(B_k) + max(-y(B_k) : y \in P(f)) = x(B_k) - min(y(B_k) : y \in P(f)) \geq 0$. The new s also can be generated by the greedy algorithm, so this step can be repeated until each $t_j \leq 2$. Now let $A = \{j : t_j = 0\}$ and let $B = \{j : t_j = 2\}$. Then the dual greedy algorithm puts $s_{A,B} = 1$ and all other values of s are zero. So the optimal value of (4.8) is just $f(A, B) + 2x(B) + x(V \setminus (A \cup B))$, which is the desired value. ∎

Now the min-max Theorem 2.6 follows from Theorems 4.6 and 2.3. In order to prove the more general Theorem 2.7 by these methods we need to do two things. First, we need to show that, where $f(A, B)$ denotes $\max(x(A) - x(B) : x$ is the degree sequence of a u-capacitated b-matching), f is bisubmodular. To prove this, it is enough to show, for sets A, B, A', B' such that $(A \cup A') \cap (B \cup B') = \emptyset$, there is a u-capacitated b-matching whose degree sequence x satisfies

$$x(A) - x(B) = f(A, B);$$
$$x(A \cap A') - x(B \cap B') = f(A \cap A', B \cap B');$$
$$x(A \cup A') - x(B \cup B') = f(A \cup A', B \cup B').$$

(Then (4.3) follows from the fact that $x(A') - x(B') \leq f(A', B')$.) Such a b-matching can be constructed by modifying an algorithm for finding a b-matching of maximum component-sum. The same algorithm can be used to give a formula for $f(A, B)$ and thus to do the second thing, namely, reduce the expression in Theorem 4.6 to the one in Theorem 2.7. We do not go into the details.

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